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Technical Note

Validation of Navier–Stokes equations for slip flow analysis within transition region

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ABSTRACT

Navier–Stokes equations with first-order and second-order accurate slip boundary conditions are provided to describe the two-dimensional gaseous steady laminar flow between two plates in transition region. A new similarity transformation for the controlling equations with boundary conditions was given so as to obtain the ordinary differential controlling equation. An analytical solution was provided for this strong nonlinear ordinary differential equation using a powerful, easy-to-use analytic technique for nonlinear problems, that is, the homotopy analysis method (HAM). By analyzing solutions and comparing with the results of other investigators, it is proved that the results are not accurate or physically wrong either obtained from calculation of Navier–Stokes equations with first-order boundary condition or second-order slip boundary condition in transition flow region, although some parameters may fit well with the experimental data. So, Navier–Stokes equations cannot govern the gaseous flow in transition region.

1. Introduction

There is an increasing number of applications that involve micro-scale devices including sensor, heat exchangers and micro-power system. Micro-system will own many advantages that do not appear in conventional size with the size decreasing [1]. Micro-channels, however, are the basic structures in these systems. Based on Knudsen (Kn) number, Beskok [2] classified the gas flow in micro-channels into four flow regimes: continuum flow regime ($Kn \leqslant 0.001$), slip flow regime ($0.001 < Kn \leqslant 0.1$), transition flow regime ($0.1 < Kn \leqslant 10$) and free molecular flow regime ($Kn > 10$). The flow in many application of the micro/nano systems (the characteristic length is ranged from 0.2 to 20 μm), such as hard disk drive, micro-pumps, micro-valves and micro-nozzles, is in slip and transition flow regime, which is characterized by slip flow at wall. The difficulties in experimental instruments and making precise measurements restrict the developments of experimental methods as the size decrease. Till now, only a few researchers [3,4] used the experimental methods to investigate these problems. Numerical simulations or analytical solutions have been the more commonly used approaches to study gas flows through micro-channels.

Many researchers believed that rarefaction effects become more important, and eventually the continuum approach breaks down altogether as Kn number increases. Theoretically, the Navier–Stokes equations are first-order accurate in Kn. So, they insisted that Navier–Stokes equations cannot govern slip flow in transition region with second-order or higher-order slip model. Namely, traditional macroscopic hydrodynamic descriptions are invalid, and the full Boltzmann equation, Burnett equations and the molecular-based methods such as molecular dynamics (MD) and direct simulation Monte Carlo (DSMC) [5] have to be employed for analytical or numerical studies. DSMC [6,7] is the main tool for the research, and some good results were obtained by using this method. The LBM has received considerable attention by fluid dynamic researchers [8–11] in recent years. They insisted that the results obtained from their calculation fit well with the experimental data. Recently, Niu et al. [12] proposed a thermal lattice Boltzmann BGK model for simulating the micro-thermal flow. The obtained results were found to be in good agreement with those given from the DSMC, the MD approaches and the Maxwell theoretical prediction.

On the other hand, some investigators [13,14,16] insisted that using Navier–Stokes equations to describe rarefied gases flow in transition region can still be valid by using a high order slip boundary condition. Recently, Dongari et al. [14] used a second-order accurate slip boundary condition that was suggested by Sreekanth [15] to predict the flow characteristics. The results shown that the validity of Navier–Stokes to rarefied gases can possibly be increased up to $Kn = 5$.

The motivation of this paper is to lucubrate whether Navier–Stokes equations with first-order or second-order accurate slip boundary conditions can govern the high Kn number flows by using an analytic method – homotopy analysis method (HAM) [17–23].

2. Slip model

The classical description of the velocity slip in rarefied gases flowing over a solid surface is the Maxwell [24] slip condition, and is widely implemented in current rarefied gas flow researches.

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The ideal gas flow is considered and it is assumed that a percentage of the wall collisions were specular whereas a remained percentage were diffuse. Such an assumption allows an exchange of momentum between the gas and the wall. The corresponding velocity slip is given by,

$$u_s - u_w = \lambda \frac{\partial u}{\partial y} \tag{1}$$

Here, u_s is slip velocity; u_w is wall velocity, $\lambda = \frac{2-\sigma_v}{\sigma_v} l$. l is molecular mean free path; σ_v is the tangential accommodation coefficient.

Based on Maxwell's first-order slip model, Thompson developed a second-order slip model,

$$u_s - u_w = \frac{2-\sigma_v}{\sigma_v}\left(Kn\frac{\partial u}{\partial y} + \frac{Kn^2}{2}\frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

Many researches reported that this model cannot predict the flow in high Kn number. So Beskok and Karniadakis [13] then suggested the improved second-order slip conditions,

$$u_s - u_w = \frac{2-\sigma_v}{\sigma_v}\frac{Kn}{1-bKn}\frac{\partial u}{\partial y} \tag{3}$$

where b is an empirical parameter whose value can be determined by DSMC simulations for various Knudsen number regimes. They claimed that they obtained excellent results by utilizing this model as boundary conditions with Navier–Stokes equations in transition region even in free molecular region.

For more widely range, Sreekanth [15] suggested the following general form of second-order slip model,

$$u_s - u_w = C_1 l \frac{\partial u}{\partial y} + C_2 l^2 \frac{\partial^2 u}{\partial y^2} \tag{4}$$

Here C_1 and C_2 are two independent coefficients named the slip coefficients.

In this paper, this general second-order slip model was used for analytical solution.

3. Mathematical description of the problem

The calculated model is shown in Fig. 1. The problem is assumed to be two-dimension. The inlet velocity distribution is uniform. The distance between the two plates is $2H$. The mathematical problems are derived based on the following assumptions: (1) the governing equations based on Navier–Stokes equations with slip flow at walls can be used to describe the physical processes; (2) the process is two-dimensional steady flow; (3) the flow is laminar; (4) the body forces are neglected; (5) the effect of compressibility and viscosity heating are neglected. According to the above assumptions, the Navier–Stokes equations with the slip flow boundary between two parallel plates in micro-scale are given as follows:

$$\text{Continuity equation} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5}$$

$$\text{Momentum equation} \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{6}$$

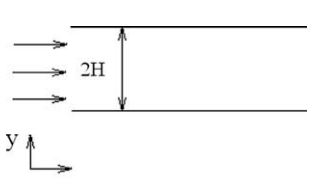

Fig. 1. A schematic diagram of microchannel.

Boundary conditions are

On the walls

$$\text{First-order slip model} \quad y = 0 \quad u = \lambda\frac{\partial u}{\partial y} \tag{7a}$$

$$\text{Second-order slip model} \quad y = 0 \quad u = C_1\lambda\frac{\partial u}{\partial y} + C_2\lambda\frac{\partial^2 u}{\partial y^2} \tag{7b}$$

$$\text{On the centerline} \quad y = H \quad \frac{\partial u}{\partial y} = 0 \quad v = 0 \tag{8}$$

Here, $\lambda = \frac{2-\sigma_v}{\sigma_v} l$; l is molecular mean free path; σ_v is the tangential accommodation coefficient. In this paper, it is defined $\sigma_v = 1$.

It is convenient to introduce the following dimensionless variables,

$$\eta = \frac{y}{H}, \quad \varphi(\eta) = \left[A + B\frac{x^\sigma}{H}\right] f(\eta), \quad u = \frac{\partial \varphi}{\partial y}, \quad v = -\frac{\partial \varphi}{\partial x} \tag{9}$$

Defining:

$$\sigma = 1 \quad p = \left[Ax + \frac{B}{2H}x^2 + C\right]k \tag{10}$$

where A, B, C, K are constant. Substituting Eqs. (9),(10) into Eqs. (5), (6), (7), (8) the problem reduces to an ordinary differential equation and boundary conditions as follow,

$$\nu f''' + Bff'' - B(f')^2 = \frac{H^3}{\rho}k \tag{11}$$

$$\text{First-order slip model} \quad \lambda f''(0) - Hf'(0) = 0 \tag{12a}$$

$$\text{Second-order slip model} \quad H^2 f'(0) - C_1\lambda^2 f'''(0) - C_2\lambda H f''(0) = 0 \tag{12b}$$

and

$$f''(1) = 0 \quad f(1) = 0 \tag{13}$$

As we know, $Kn = \frac{l}{H}$, and then Eq. (12) could be

$$Knf''(0) - f'(0) = 0 \tag{14a}$$

$$f'(0) - C_1 Kn^2 f'''(0) - C_2 Kn f''(0) = 0 \tag{14b}$$

4. Mathematical analyses for HAM solution

4.1. Basic idea of HAM

In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, which is a powerful, easy-to-use analytic technique for nonlinear problems, namely homotopy analysis method (HAM) [17–23]. This method has been successfully applied to many types of nonlinear problems by others [25–29]. The basic idea of HAM is introduced in Liao [17].

4.2. Solve with HAM

4.2.1. Solve first-order slip model

The homotopy analysis method is applied to solve the set of nonlinear Eq. (11). According to Eqs. (13) and (14a) it is convenient to choose,

$$f_0(\eta) = \frac{1}{6}\eta^3 - \frac{1}{2}\eta^2 - \eta Kn + Kn + \frac{1}{3} \tag{15}$$

Besides, choosing

$$L[\Phi(\eta, q)] = \frac{\partial^3 \Phi(\eta, q)}{\partial \eta^3} \tag{16}$$

as the auxiliary linear operator. Based on Eq. (11), defining the nonlinear operators

$$N[\Phi(\eta,q)] = v\frac{\partial^3\Phi(\eta,q)}{\partial\eta^3} + B\Phi(\eta,q)\frac{\partial^2\Phi(\eta,q)}{\partial\eta^2} - B\left[\frac{\partial\Phi(\eta,q)}{\partial\eta}\right]^2 - \frac{H^3}{\rho}k \tag{17}$$

where $q \in [0,1]$ is an embedding parameter; $\Phi(\eta,q)$ is real functions of η and q, then, the so-called zero-order deformation equation were obtained,

$$(1-q)[\Phi'''(\eta,q) - f_0'''(\eta)] = q\hbar N[\Phi(\eta,q)] \tag{18}$$

Boundary conditions,

$$Kn\Phi''(0,q) - \Phi'(0,q) = 0, \quad \Phi(0,q) = 0, \quad \Phi'(1,q) = 0 \tag{19}$$

Obviously, $\Phi(\eta,0) = f_0(\eta)$, $\Phi(\eta,1) = f(\eta)$. Differentiating the zero-order deformation (18) m times with respect to q, then dividing it by m!, and finally setting $q = 0$, the mth-order deformation equations were gotten,

$$L[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar R_m(f_{m-1}) \tag{20}$$

With boundary conditions

$$Knf''_m(0) - f'_m(0) = 0, f_m(1) = 0, \quad f''_m(1) = 0 \tag{21}$$

Here

$$\chi_m = \begin{cases} 0 & m \leqslant 1 \\ 1 & m \geqslant 2 \end{cases} \tag{22}$$

$$R_m(f_{m-1}) = vf'''_{m-1}(\eta) - \frac{kH^3}{\rho}(1-\chi_m) + \sum_{k=0}^{m-1}[Bf_k(\eta)f''_{m-1-k}(\eta) - Bf'_{m-1-k}(\eta)f'_k(\eta)] \tag{23}$$

Substitute Eq. (23) into Eq. (20), there are

$$f'''_m(\eta) - \chi_m f'''_{m-1}(\eta) = \hbar\left\{vf'''_{m-1}(\eta) - \frac{H^3k}{\rho}(1-\chi_m) + \sum_{k=0}^{m-1}[Bf_k(\eta)f''_{m-1-k}(\eta) - Bf'_{m-1-k}(\eta)f'_k(\eta)]\right\} \tag{24}$$

4.2.2. Solve second-order slip model

According to (13) and (14b), it is convenient to choose,

$$f_0(\eta) = \frac{1}{6}\eta^3 - \frac{1}{2}\eta^2 + (C_1Kn + C_2Kn^2)\eta - C_2Kn^2 - C_1Kn + \frac{1}{3} \tag{25}$$

It has the same auxiliary linear operator and nonlinear operators as defined in the process of solving the first-order slip model. So the recursive formula must be the same, and the solved processes are almost identical. The difference only exists in boundary conditions. Boundary conditions (21) for first-slip model can be written as follow for second-order model,

$$f'_m(0) - C_1Kn^2f'''_m(0) - C_2Knf''_m(0) = 0 \quad f_m(1) = 0 \quad f''_m(1) = 0 \tag{26}$$

The symbolic calculation software Mathematica was used to solve the set of linear differential Eq. (20) with conditions (21) and (26) up to first few orders of approximations.

The corresponding second-order approximations can be expressed by,

$$f(\eta) = f_0(\eta) + f_1(\eta) + f_2(\eta) \tag{27}$$

4.3. Convergence theorem

For first-order slip model, the series $f(\eta) = f_0(\eta) + \sum_{m=1}^{+\infty} f_m(\eta)$ is an exact solution of Eqs. (11), (13) and (14) as long as it is convergent. Here, $f_m(\eta)$ satisfies Eqs. (18), (19) and definitions (22), (23) are correct. The details of proof process of convergence theorem for first-order slip model can be found in the paper [29]. It is almost the same way to prove convergence theorem for second-order slip model.

5. Results analysis

It is worth noting to point out that the advantage of the homotopy analysis method shows a fast convergence of the solutions by means of the auxiliary parameter, $\hbar \neq 0$. To adjust and control the convergence region of solution series [17], the auxiliary parameter for both first-slip model and second-slip model is chosen as $\hbar = 1$.

In the calculation, we use first few orders to approximate the series solutions. Table 1 shows the f value for first few orders at given Kn number ($Kn = 0.024$ in slip flow region, $Kn = 0.24$ in transition region). As is shown, f_3 are small enough to be neglected both for first-order slip model and second-order slip model. The higher order of f are too smaller than f_3 to be neglected. So, the assumption $f(\eta) = f_0(\eta) + f_1(\eta) + f_2(\eta)$(Eq. (27)) utilized in the current calculation is reasonable. So the accurate enough results can be obtained.

Fig. 2 shows fully developed velocity profiles at different Knudsen number in slip flow region. Here, band k is calculated from the pressure distribution along flow direction in Barber and Emerson's study [30]. Velocity profiles fits well with that obtained by Barber and Emerson [30], seeing Fig. 2. That is to say, Navier–Stokes equations with first-order slip model can describe flow in slip region accurately.

For higher Kn number ($Kn > 0.1$, in transition flow region), both first-order slip model and second-order slip model were selected to calculate the slip flow. Fig. 3 shows the mass flow rate variation with pressure ratio in high Kn number region. According to Fig. 3, the results calculated from second-order slip model fit very well with experimental data obtained by Arkilic [3]. However, much error exists by using first-order slip model. These results are almost the same as that obtained by Dongari et al. [14]. It can be concluded that first-order slip model is not suitable in govern high Kn number region flow.

According to the pressure distribution at different Kn number along the channel measured by Hsieh [4], parameters A, B, C and k were obtained. Furthermore, the corresponding Re numbers were calculated. Table 2 shows the comparison of Re number calculated by using second-order slip model in this research with the experimental data and their theoretical results using first-order slip model. Almost the same conclusion can be made as that above. That is, the data obtained by solving second-order slip model are much more accurate than that obtained by solving first-order slip model. Error for second-order slip model were all below 7%, however, the errors of the first-order slip model were almost exceed 10%. In another words, Navier–Stokes equations with first-order slip boundary condition cannot govern high Kn number flow.

Table 1
f Value for first few orders at given Kn number

f_m	First-order slip model		Second-order slip model	
	$Kn = 0.024$	$Kn = 0.24$	$Kn = 0.024$	$Kn = 0.24$
F_0	0.267815	0.390934	0.238129	0.0652487
F_1	0.0678415	0.0001981	0.000136539	0.000192004
F_2	−0.000504084	−2.95717E−9	−2.02814E−09	−2.84559E−9
F_3	3.43016E−06	3.68159E−14	2.75715E−14	3.57633E−14
F_4	−2.08621E−08	−3.34607E−19	−3.34292E−19	−3.3661E−19
F_5	1.06355E−10	7.14115E−25	3.38E−24	1.12457E−24
F_6	−3.6492E−13	5.36636E−29	−2.26555E−29	4.22465E−29
F_7	−4.83781E−16	−1.46596E−33	−7.67555E−35	−1.26837E−33

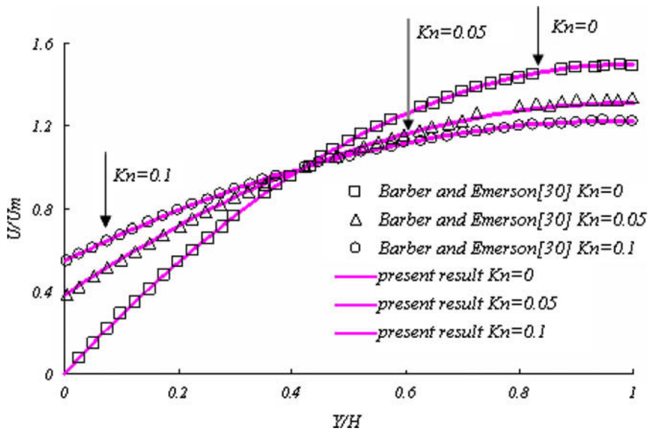

Fig. 2. Fully developed velocity profiles in slip flow region.

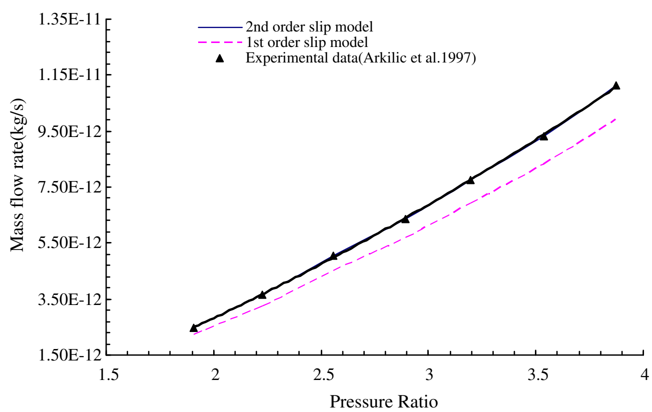

Fig. 3. Mass flow rate variation with pressure ratio ($C_1 = 1.466$, $C_2 = 0.9756$ [15]).

Table 2
Comparison of *Re* number obtained by Hsieh [4] with that obtained by us ($C_1 = 1.466$, $C_2 = 0.9756$ [15])

Re			Error (%)	
Experimental data [4]	First-order model [4]	Second-order by HAM	First-order model [4]	Second-order by HAM
89.4	100.6	83.2	12.53	6.96
67.4	76.4	68.7	13.35	1.96
50.7	56.6	51.1	11.64	0.74
32.4	35.8	30.9	10.49	4.65
16.1	17.2	14.8	6.83	8.04
5.7	6	5.6	5.26	1.13
2.6	2.9	2.6	11.54	0.71

However, *Re* numbers fit well with experimental data by utilizing second-order slip boundary condition and proper choose of the slip coefficients.

Based on these results, one may conclude that Navier–Stokes equations with second-order slip boundary conditions for rarefied gases flow can possibly be valid by proper choice of the slip coefficients.

Fig. 4 shows the velocity profiles at different *Kn* number. Here, second-order slip model was used for calculation, and the same value of the slip coefficients $C_1 = 1.466$, $C_2 = 0.9756$ [15] as above were used. As it shows, the slip velocity (the velocity at wall ($Y/H = 0$)) increases with the increasing of *Kn* number. When *Kn* number goes up to 0.75, the slip velocity reaches maximum. Till now, the velocity profiles and the variation trend of slip velocity are physically reasonable. However, as *Kn* number increases further, the slip velocity begins to decrease. The value of slip velocity equals to zero when *Kn* number reaches about 1.5, here the velocity profiles are the same as that when $Kn = 0$ as it shows in the figure. It is wrong physically, because, as we have known, the slip velocity increases with the increasing of *Kn* number. Even more, when *Kn* number is larger than 1.5, the velocity profiles appears reversal. That is, the slip velocity is even larger than the centerline velocity. Afterward, when *Kn* number increases, the trend of velocity profiles become flat. These phenomena are incompatible with the inner flow mechanism. In another word, it is physically wrong. So, although some physical parameters such as mass flow rate and *Re* number obtained by solving Navier–Stokes equations with second-order slip boundary condition and proper choosing of the slip coefficients C_1 and C_2 fit well with experimental data, the model still cannot be used to govern flow in transition region as the velocity profiles at high *Kn* number are physically wrong.

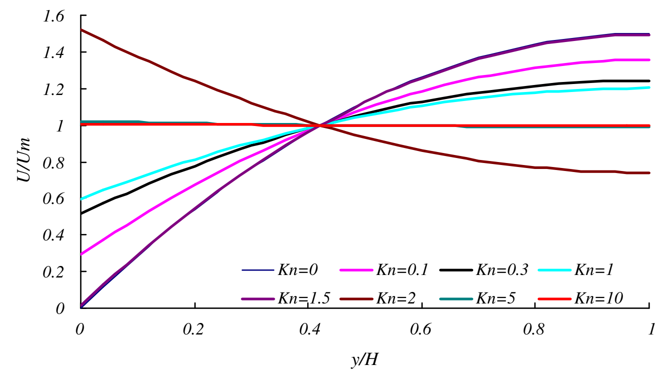

Fig. 4. Velocity profiles with second-order slip model at different *Kn* number ($C_1 = 1.466$, $C_2 = 0.9756$ [15]).

6. Conclusions

In this paper, two-dimensional slip flow between two parallel plates was analytically investigated. Ordinary differential equation that describes this problem was obtained by using a new similarity transformation. The homotopy analysis method (HAM) was applied to obtain the solution of this strongly nonlinear ordinary differential equation.

It can be concluded that Navier–Stokes equations with first-order slip model can govern flow in slip region accurately. However, much error exists by using first-order slip model in transition region. So first-order slip model is not suitable to govern high *Kn* number region flow. Some physical parameters such as mass flow rate and *Re* number obtained by solving Navier–Stokes equations with second-order slip boundary condition and proper choosing of the slip coefficients C_1 and C_2 fit well with experimental data. But the model still cannot be used to govern flow in transition region as the velocity profiles at high *Kn* number are physically wrong.

On the basis of above analyze, Navier–Stokes equations with current slip model boundary condition are not suitable for description of flow in transition region.

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